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Linear systems with adiabatic fluctuations*

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Abstract. We consider a dynamical system subjected to weak but adiabatically slow fluctuations of an external origin. Based on the 'adiabatic following' approximation we carry out an expansion in $\alpha|\mu|^{-1}$, where α is the strength of fluctuations and $|\mu|^{-1}$ refers to the time scale of evolution of the unperturbed system to obtain a linear differential equation for the average solution. The theory is applied to the problems of a damped harmonic oscillator and diffusion in a turbulent fluid. The result is the realization of a 'renormalized' diffusion constant or damping constant for the respective problems. The applicability of the method has been analysed critically.

1. Introduction

The standard paradigm of the temporal evolution of nonequilibrium processes, regarded, in general, as stochastic processes, is the century-old problem of Brownian motion [1, 2]. This involves the random motion of microscopic particles effectively introducing the motion of a physical system, the Brownian particle to be observed on a macroscopic level. To generate the successive levels of description from the microscopic to the macroscopic realm one essentially introduces coarse-graining of space and time in the dynamics. Although there exists no general program of coarse-graining, it is nevertheless possible to realize the dynamics of stochastic processes in terms of some systematic separation of time scales consistent with the experiments at the macroscopic level of description.

The standard separation of time scales in the description of Brownian motion involves correlation functions which are nonzero over some interval τ_c, which is the correlation time of fluctuations and we require that Δt, the coarse-grained time scale over which one observes the average motion, is much greater than τ_c, such that $\gamma^{-1} \gg \Delta t \gg \tau_c$, where γ^{-1} is the system's damping time. Physically this implies that one smooths out the fluctuations of the system on a time scale during which microscopic particles are correlated but not on a scale during which the system is damped. Thus the fluctuations considered in the stochastic process of the Brownian motion are weak and rapid.

In the present problem we consider a multivariate dynamical system driven by weak but adiabatically slow fluctuations. The slow fluctuations characterized by very long correlation time have also attracted a lot of attention of various workers over the years [2–4, 7, 8, 11, 14, 15]. While the overwhelming majority of the treatment of stochastic differential equation with fast fluctuations is based on the assumption that there is a very short autocorrelation time τ_c, such that one can adopt the scheme of expansion in $\alpha\tau_c$, a suitable simplifying approximation for dealing with very long correlation times is

* This paper is dedicated to Professor Mihir Chaudhury on the occasion of his 60th birthday.

relatively scarce. In general, the problem of long correlation time is handled theoretically at the expense of a severe restriction on the type of stochastic behaviour. For instance, several authors [2–4, 7, 8, 11, 14, 15] have tried the linear and nonlinear models within the framework of Markov processes of the dichotomic processes type, two-state Markov processes, random telegraphic processes, etc. Our aim here is to explore a *perturbative method* for finding an equation for the average solution pertaining to the separation of time scale implied in the inequality

$$\frac{1}{|\mu|} \ll \Delta t \ll \tau_c$$

where $|\mu|$ is the largest eigenvalue of the unperturbed system, *where we do not keep any restriction on the type of stochastic behaviour*. The strategy being perturbative is based on an expansion in $\alpha|\mu|^{-1}$ rather than in $\alpha\tau_c$ as is done in the case of fast fluctuations. The method dealt with in the present treatment is thus somewhat complementary to the scheme of expansion of the latter kind.

To put the issue in a proper perspective we first borrow a simple example of adiabatic dynamics in terms of Bloch equations [5, 6], well known in magnetic resonance and quantum optical experiments. The problem concerns a two-level system interacting with a single-mode electromagnetic field, where the field $\mathcal{E}(t)$ varies slowly enough 'adiabatically' on the time scale of the inverse of the damping constant or frequency detuning between the atom and the field. The term 'adiabatic following' is thus used to describe collectively the associated experimental phenomena [5, 6]. The model is described by the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -1/T_2 & -\Delta & 0 \\ \Delta & -1/T_2 & g\mathcal{E}(t) \\ 0 & -g\mathcal{E}(t) & -1/T_1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w_{\mathrm{eq}}/T_1 \end{pmatrix}. \qquad (1)$$

Here u, v, w are the Bloch vector components, T_1 and T_2 are the energy and dephasing relaxation terms, Δ is the detuning of the frequency of the field $\mathcal{E}(t)$ from that of the two-level system. g includes the effect of coupling of the atom to the field. The equilibrium value, towards which the population inversion w relaxes when $\mathcal{E} = 0$ is denoted by w_{eq}. Adiabatic following approximation asserts that if the field $\mathcal{E}(t)$ is varied slowly enough then w, the population inversion variable would follow adiabatically from -1 to $\sim +1$ in the process, i.e. a ground state population could be adiabatically inverted.

Our problem in the present investigation concerns such processes where the adiabatic variation of $\mathcal{E}(t)$, in addition, is stochastic. Thus the usual limit in the 'adiabatic following' applies, i.e. the rate of variation of the pulse or fluctuations is much small compared to the relaxation rate of the system. With these in mind we may treat equation (1) as a stochastic differential equation provided the stochastic properties of $\mathcal{E}(t)$ are *a priori* known.

To formulate the problem we thus consider a system subject to fluctuating external forces where the fluctuations are weak and adiabatically slow. The equation of motion then become a stochastic differential equation, a particular category of which is a general form of equation (1) (for simplicity we disregard the constant part on the right-hand side),

$$\dot{u} = \boldsymbol{A}(t)\, u. \qquad (2)$$

Here $\boldsymbol{A}(t)$ is a random function of time, stochastic properties of which are given. Linear multiplicative noise (equation (2)) has wide applications in studying the random Markov process [7], fluctuating barrier crossing [8], enzymatic kinetics in biology [9], nuclear magnetic resonance in physics [10] and stochastic resonance in linear systems [11] and in many other contexts [12].

Based on the systematic separation of time scales using the adiabatic following approximation, a differential equation for the average solution $\langle u \rangle$ is obtained. This approximation allows us an expansion in $\alpha|\mu|^{-1}$, where α is a measure of the strength of fluctuations and μ^{-1} refers to the internal time scale of the unperturbed system.

As an immediate application of the method we treat the problem of a damped harmonic oscillator with adiabatically fluctuating frequency. The method is extended to the problem of diffusion in a turbulent fluid as another illustration. The central result is that one realizes a 'renormalized' transport coefficient or a damping constant so that the diffusion or the damping process gets significantly modified by adiabatic stochasticity. We show that the method is equipped to deal with similar kinds of stochastic processes.

The outlay of the paper is as follows. In section 2 we discuss the method of adiabatic following approximation on the stochastic differential equation of the form (2). The essential idea is to extract the average dynamics of the relevant physical quantity. The method has been analysed critically in section 3. The method is applied to two specific cases in section 4. We point out that a wide class of problems can be treated in a similar way. The paper is concluded in section 5.

2. A method for weak and adiabatic fluctuations

To start with we consider a linear equation of the type (2) and rewrite it as

$$\dot{u} = \{\boldsymbol{A}_0 + \alpha \boldsymbol{A}_1(t)\}u \tag{3}$$

where u is a vector with n components, $\boldsymbol{A}_0$ is a constant matrix of dimension $n \times n$ and $\boldsymbol{A}_1(t)$ is a stochastic matrix, α is a parameter (of dimension $1/t$) which measures the strength of fluctuation.

It is convenient to assume that $\boldsymbol{A}_1(t)$ is a stationary process with $\langle \boldsymbol{A}_1(t) \rangle = 0$. Equation (3) admits of two important time scales of the system measured by the inverse of the largest eigenvalue of the matrix $\boldsymbol{A}_0$ and the time scale of fluctuations of $\boldsymbol{A}_1(t)$ (the correlation time of fluctuation). As has already been mentioned in the treatment of the overwhelming majority of stochastic processes, such as, motion of a Brownian particle in a fluid or electromagnetic waves in a turbulent atmosphere, one essentially considers a situation where the fluctuations are weak and rapid. The correlation time of fluctuations is much shorter compared to the time scale set by the inverse of the eigenvalues of $\boldsymbol{A}_0$. In the appropriate limit we encounter the delta-correlated events and solve approximately or exactly the relevant stochastic differential equations [2]. The familiar examples of paramagnetic resonance and line broadening are well known in this context.

Since in the present problem we consider a stochastic process in which the fluctuations are weak but adiabatically slow, $\boldsymbol{A}_1(t)$ is an adiabatic stochastic process. Therefore the usual procedure of systematic cumulant expansion, which inherently takes into account the short correlation time of fluctuations, is not valid. An alternative treatment is thus sought for.

To this end we first introduce an interaction representation as given by

$$u(t) = \exp(\boldsymbol{A}_0 t)\, v(t)$$

and applying it to equation (3) we obtain

$$\dot{v} = \alpha\, \boldsymbol{V}(t)\, v$$

where

$$\boldsymbol{V}(t) = \exp(-\boldsymbol{A}_0 t)\, \boldsymbol{A}_1(t) \exp(\boldsymbol{A}_0 t).$$

On integration the last equation yields

$$v(t) = v(0) + \alpha \int_0^t \boldsymbol{V}(t')\, v(t')\, \mathrm{d}t'. \tag{4}$$

On iterating equation (4) once, we are now led to an ensemble average equation of the following form:

$$\langle v(t)\rangle = v(0) + \alpha^2 \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \,\langle \boldsymbol{V}(t')\, \boldsymbol{V}(t'')\, v(t'')\rangle. \tag{5}$$

The equation is still exact since no second-order approximation (as is usually done) has been used.

Now taking the time derivative of $v(t)$ we arrive at the following integrodifferential equation in which the initial value $v(0)$ no longer appears,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(t)\rangle = \alpha^2 \int_0^t \langle \boldsymbol{V}(t)\, \boldsymbol{V}(t')\, v(t')\rangle\, \mathrm{d}t'. \tag{6}$$

Making use of a change of integration variable $t' = t - \tau$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(t)\rangle = \alpha^2 \int_0^t \langle \boldsymbol{V}(t)\, \boldsymbol{V}(t-\tau)\, v(t-\tau)\rangle\, \mathrm{d}\tau. \tag{7}$$

Reverting back to the original representation equation (7) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \boldsymbol{A}_0\langle u\rangle + \alpha^2 \int_0^t \langle \boldsymbol{A}_1(t)\, \exp(\boldsymbol{A}_0\tau)\, \boldsymbol{A}_1(t-\tau)\, u(t-\tau)\rangle\, \mathrm{d}\tau. \tag{8}$$

The adiabatic following assumption, that $\boldsymbol{A}_1(t)$ and the components of $u(t)$ vary slowly on the scale of inverse of $\boldsymbol{A}_0$, can now be utilized. Following Crisp [6] we note that a Taylor series expansion of $\boldsymbol{A}_1(t-\tau)\, u(t-\tau)$ in the average $\langle\cdots\rangle$ of the α^2-term in equation (8) allows the integral to be evaluated and the last equation reduces to the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \boldsymbol{A}_0\langle u\rangle + \alpha^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \boldsymbol{A}_1(t) \left\{ \int_0^\infty \mathrm{d}\tau\, \exp(\boldsymbol{A}_0\tau)\tau^n \right\} \frac{\mathrm{d}^n}{\mathrm{d}t^n}[\boldsymbol{A}_1(t)\, u(t)] \right\rangle. \tag{9}$$

The integral in equation (9) can be evaluated by rewriting it in terms of the following matrix elements:

$$\begin{aligned} I_{ik}^n &= \int_0^\infty \mathrm{d}\tau\, \tau^n \sum_j D_{ij}\, \mathrm{e}^{\mu_{jj}\tau} D_{jk}^{-1} \\ &= \sum_j D_{ij} \frac{n!}{(\mu_{jj})^{n+1}} D_{jk}^{-1} \qquad \mathrm{Re}\,\mu_{jj} < 0. \end{aligned}$$

Therefore,

$$\boldsymbol{I}^n = n!\boldsymbol{D}\boldsymbol{E}_{n+1}\boldsymbol{D}^{-1} \tag{10}$$

where $\boldsymbol{D}$ is a matrix which diagonalizes $\boldsymbol{A}_0$ and

$$\boldsymbol{E}_{n+1} = \begin{pmatrix} 1/\mu_{11}^{n+1} & & 0 \\ & \ddots & \\ 0 & & 1/\mu_{jj}^{n+1} \end{pmatrix}$$

and μ_{jj} are the eigenvalues of $\boldsymbol{A}_0$.

Equation (9) then assumes the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \boldsymbol{A}_0\langle u\rangle + \alpha^2 \sum_{n=o}^{\infty} (-1)^n \left\langle \boldsymbol{A}_1(t)\, \boldsymbol{D}\boldsymbol{E}_{n+1}\boldsymbol{D}^{-1} \frac{\mathrm{d}^n}{\mathrm{d}t^n}[\boldsymbol{A}_1(t)\, u(t)] \right\rangle. \tag{11}$$

Although the equation involves an infinite series it is expected to yield a useful result in the adiabatic following approximation. If this approximation is valid, the quantity $[\boldsymbol{A}_1(t)\,u(t)]$ varies little in the time $\left|\mu_{jj}^{n+1}\right|^{-1}$ of $\boldsymbol{E}_{n+1}$ and the series converges rapidly. Keeping only the two lowest-order terms we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \boldsymbol{A}_0\langle u\rangle + \alpha^2\langle \boldsymbol{A}_1(t)\,\boldsymbol{X}_1\,\boldsymbol{A}_1(t)\,u(t)\rangle - \alpha^2\langle \boldsymbol{A}_1(t)\,\boldsymbol{X}_2\,\dot{\boldsymbol{A}}_1(t)\,u(t)\rangle$$
$$-\alpha^2\langle \boldsymbol{A}_1(t)\,\boldsymbol{X}_2\,\boldsymbol{A}_1(t)\,\dot{u}(t)\rangle \tag{12}$$

where

$$\boldsymbol{X}_{n+1} = \boldsymbol{D}\boldsymbol{E}_{n+1}\boldsymbol{D}^{-1}. \tag{13}$$

It is evident that the average $\langle\dot{u}\rangle$ is related to a more complicated average. As a next approximation [13–15] we now suppose that the latter averages may be broken up as

$$\langle \boldsymbol{A}_1(t)\,\boldsymbol{X}_1\,\boldsymbol{A}_1(t)\,u(t)\rangle \approx \langle \boldsymbol{A}_1(t)\,\boldsymbol{X}_1\,\boldsymbol{A}_1(t)\rangle\langle u(t)\rangle \tag{14}$$

and so on. Keeping only the terms up to the order of α^2 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \big\{\boldsymbol{A}_0 + \alpha^2[\langle \boldsymbol{A}_1(t)\,\boldsymbol{X}_1\,\boldsymbol{A}_1(t)\rangle - \langle \boldsymbol{A}_1(t)\,\boldsymbol{X}_2\,\dot{\boldsymbol{A}}_1(t)\rangle$$
$$-\langle \boldsymbol{A}_1(t)\,\boldsymbol{X}_2\,\boldsymbol{A}_1(t)\rangle\boldsymbol{A}_0]\big\}\langle u(t)\rangle. \tag{15}$$

Thus the average of $u(t)$ obeys a nonstochastic differential equation in which the effect of weak adiabatic fluctuations is accounted for by 'renormalizing' $\boldsymbol{A}_0$ through the addition of constant terms of the order of α^2. The net effect is that depending on the specificity of the situations one realizes a dissipative or a gain term in the average dynamics. We note in passing that the average dynamics of u is independent of any explicit correlation function.

3. Discussions on the method

The theory of stochastic differential equations with multiplicative noise has a long history. The stochastic processes dealt with the overwhelming majority of the cases concerning the fast processes (more precisely, the correlation time between the noise events has been considered to be the shortest time scale of the dynamics). In the previous section we have considered a stochastic process which is adiabatically slow. The traditional scheme of solving stochastic differential equations with fast noise processes is that one reduces them to Bourret's integral equations [13] and then performs the decoupling of the product of operators. Here we have followed the equation scheme up to equation (9) and then make use of the 'adiabatic following' approximation. It is necessary to make the following distinctions.

First, note that in going from equation (8) to (11) we have made no approximation so far as the full infinite series is concerned. Also each term is not of order $\alpha\tau_c$ as in the case of fast processes (as emphasized by van Kampen [14]), but of order $\alpha\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\boldsymbol{A}(t)\,u(t)]/\mu_{jj}^{n+1}$. Just as the theory of fast processes is valid for $\alpha\tau_c$ very small which implies that the successive cumulants in the expansion are small, validity of the description of adiabatic processes rests on the smallness of successive $\alpha\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\boldsymbol{A}(t)\,u(t)]/\mu_{jj}^{n+1}$ terms. Thus the two expansions are essentially different.

Second, the decoupling approximation has been carried out both in the fast as well as in the slow processes. Its justifications in the former case has been established early on by Brissaud and Frisch [15]. It had been strongly advocated by van Kampen [14] who has asserted that although it seems to neglect certain correlations, the 'statistical mechanics of transport processes would be in a very sorry state without such a hypothesis'. It is not difficult to comprehend that its spiritual root lies in 'stosszahlansatz', 'molecular chaos assumption' or 'random-phase approximation'. The essential point, however, in the decoupling scheme is the realization of a separation of time scale of the average of the product of fluctuating quantities $\boldsymbol{A}(t)$ and the average of u itself, consistent with the expansions pertaining to slow or fast processes.

That the two expansion schemes in the fast and slow stochastic processes are different can be confirmed if one compares the lowest-order terms of the corresponding evolution. According to the present scheme equation (15) itself asserts that (free motion neglected)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u\rangle \sim \frac{\alpha^2}{|\mu|}\langle u\rangle \tag{16}$$

where $|\mu|^{-1}$ refers to the time scale set by the $\boldsymbol{A}_0$ matrix which is short in the adiabatic following limit. For a fast stochastic process, on the other hand, the counterpart of equation (16) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u\rangle \sim \alpha^2\tau_c\langle u\rangle \tag{17}$$

where τ_c defines the very short correlation time of the noise [14].

The difference in the expansion schemes also makes the relative errors made in the decoupling approximation in the two cases, different. To this end we first note that equation (15) is obtained from equation (8). Up to second order it means omitting terms of the order of $(\alpha\Delta t)^3$ and higher (where Δt defines the coarse-grained time scale of the evolution of the average). As the lower bound of Δt is determined by $|\mu|^{-1}$, it implies that we neglect terms of the order of $\left(\alpha|\mu|^{-1}\right)^3$ in the evolution equation. Thus the relative error made in the decoupling approximation in the case of adiabatic expansion is $\left(\alpha|\mu|^{-1}\right)^3$.

As demonstrated by van Kampen [14] the corresponding error made in the decoupling approximation in Bourret's scheme is of the order of $(\alpha\tau_c)^3$. The workability of the decoupling approximation in the fast and slow stochastic processes is thus demonstrated in the two different expansion procedures ensuring their respective fast convergence in the limit $\alpha\tau_c$ (fast processes) or $\alpha|\mu|^{-1}$ (slow processes) small but finite.

So, to summarize, we point out that the implementation of Bourret's decoupling approximation is a major step for almost any treatment of multiplicative noise to date [2, 7, 11–15]. This is because of the fact that the average of a product of stochastic quantities does not factorize into the product of averages, although it has been argued that [2, 7, 11–21] good approximations can be derived by assuming such a factorization. In the case of fast fluctuations it has been justified if the driving stochastic noise has a fast correlation time such that Kubo number $\alpha^2\tau_c$ is very small in the cumulant expansion scheme (an expansion in $\alpha\tau_c$). The factorization has been shown to be exact in the limit of zero correlation time and in some specific noise processes [7, 14] and the solution for the average can be found exactly. The present scheme of adiabatic expansion, on the other hand, is an expansion in $\alpha|\mu|^{-1}$ and it may be argued in the same way that factorization in the slow fluctuation is valid where $\alpha^2|\mu|^{-1}$ is very small. Essentially it implies that $u(t)$ in the average (in the right-hand side of equation (12)) is realized as an average $\langle u(t)\rangle$ (which varies in the coarse-grained time scale Δt) pertaining to the separation of the time scales in the inequality $|\mu|^{-1} \ll \Delta t \ll \tau_c$ adopted in the present case instead of the inequality $\tau_c \ll \Delta t \ll |\mu|^{-1}$ employed in the case of fast fluctuation and cumulant expansion.

4. Applications

4.1. Damped harmonic oscillator with adiabatically fluctuating frequency

To illustrate the above-mentioned method we consider a model of a damped harmonic oscillator with random frequency where the fluctuation is weak and adiabatically slow on the time scale of dissipation. The opposite limit of weak and rapid fluctuations in frequency has been studied by numerous authors in connection with turbulence, wave propagation, line-broadening [10], lasers and chaotic dynamics [20, 21]. A comprehensive treatment has been given in van Kampen [14].

We are now in a position to apply the result (15) to the following equation:

$$\ddot{x} + \omega^2(t)x + \gamma\dot{x} = 0 \tag{18}$$

with an adiabatically stochastic frequency,

$$\omega^2(t) = \omega_0^2[1 + \alpha\,\xi(t)] \tag{19}$$

where $\xi(t)$ is an adiabatic stochastic process with zero mean $\langle\xi(t)\rangle = 0$; ω_0 is the frequency of the unperturbed oscillator and γ is the damping constant. α, the smallness parameter, is dimensionless in equation (19).

Rewriting equation (18) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \alpha\omega_0^2\,\xi(t)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{20}$$

one identifies,

$$\boldsymbol{A}_0 = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{pmatrix} \qquad \boldsymbol{A}_1 = \omega_0^2\,\xi(t)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad u(t) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{21}$$

of equation (15). $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are related to $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ through (13) and are given by

$$\boldsymbol{X}_1 = \begin{pmatrix} (B-A)/AB & 1/AB \\ 1 & 0 \end{pmatrix}$$
$$\boldsymbol{X}_2 = \begin{pmatrix} \left(A^2 - AB + B^2\right)/(AB)^2 & (B-A)/(AB)^2 \\ (B-A)/AB & 1/AB \end{pmatrix}$$

and

$$\boldsymbol{E}_1 = \begin{pmatrix} 1/A & 0 \\ 0 & -1/B \end{pmatrix} \qquad \boldsymbol{E}_2 = \begin{pmatrix} 1/A^2 & 0 \\ 0 & 1/B^2 \end{pmatrix}$$

where A and B are related to the eigenvalues (e_1, e_2) of the $\boldsymbol{A}_0$ matrix:

$$A = -\tfrac{1}{2}\gamma + \tfrac{1}{2}\sqrt{\gamma^2 - 4\omega_0^2} \quad (e_1)$$
$$B = \tfrac{1}{2}\gamma + \tfrac{1}{2}\sqrt{\gamma^2 - 4\omega_0^2} \quad (-e_2).$$

The D matrix is given by

$$\boldsymbol{D} = \begin{pmatrix} 1/(A+1)^{1/2} & 1/(B+1)^{1/2} \\ A/(A+1)^{1/2} & -B/(B+1)^{1/2} \end{pmatrix}.$$

So for the present problem, equation (15) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\langle x\rangle\\ \langle y\rangle\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 - \alpha^2\omega_0^2\langle\xi^2\rangle - \alpha^2\gamma\langle\xi\dot{\xi}\rangle & -\gamma + \alpha^2\langle\xi^2\rangle\gamma \end{pmatrix}\begin{pmatrix}\langle x\rangle\\ \langle y\rangle\end{pmatrix}$$

or

$$\langle\ddot{x}\rangle + \gamma\left[1 - \alpha^2\langle\xi^2\rangle\right]\langle\dot{x}\rangle + \left[\omega_0^2 + \alpha^2\omega_0^2\langle\xi^2\rangle + \alpha^2\gamma\langle\xi\dot{\xi}\rangle\right]\langle x\rangle = 0. \tag{22}$$

It is thus evident that the adiabatic fluctuations in frequency cause a suppression of the damping of the average amplitude of the oscillator. Or, in other words, the dissipative oscillator experiences a partial gain in average amplitude by an amount,

$$\gamma_{\mathrm{gain}} = \alpha^2\gamma\langle\xi^2(t)\rangle. \tag{23}$$

As expected, the frequency of the unperturbed oscillator has also undergone a shift in addition to this gain in amplitude.

The above result can be compared to the case of fast fluctuations in frequency as treated by van Kampen and others. It is interesting to note that while the adiabatic fluctuations result in a gain in amplitude, the fast fluctuations cause a damping of the average amplitude, in general. This damping may even be negative when the fluctuations are particularly strong at twice the unperturbed frequency. The latter results had been found to be useful in the context of a fluctuation–dissipation relation in chaotic dynamics [21].

The theory developed in the preceding section also permits us to calculate the dynamics of the higher moments. For example, the equations of the three moments can be found from equation (18),

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x^2\\ xy\\ y^2\end{pmatrix} = \begin{pmatrix} 0 & 2 & 0\\ -\omega^2 & -\gamma & 1\\ 0 & -2\omega^2 & -2\gamma\end{pmatrix}\begin{pmatrix}x^2\\ xy\\ y^2\end{pmatrix} \tag{24}$$

or rewriting it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x^2\\ xy\\ y^2\end{pmatrix} = \begin{pmatrix} 0 & 2 & 0\\ -\omega_0^2 & -\gamma & 1\\ 0 & -2\omega_0^2 & -2\gamma\end{pmatrix}\begin{pmatrix}x^2\\ xy\\ y^2\end{pmatrix} + \alpha\omega_0^2\xi(t)\begin{pmatrix} 0 & 0 & 0\\ -1 & 0 & 0\\ 0 & -2 & 0\end{pmatrix}\begin{pmatrix}x^2\\ xy\\ y^2\end{pmatrix} \tag{25}$$

we identify,

$$\boldsymbol{A}_0 = \begin{pmatrix} 0 & 2 & 0\\ -\omega_0^2 & -\gamma & 1\\ 0 & -2\omega_0^2 & -2\gamma\end{pmatrix} \qquad \text{and} \qquad \boldsymbol{A}_1(t) = \alpha\omega_0^2\xi(t)\begin{pmatrix} 0 & 0 & 0\\ -1 & 0 & 0\\ 0 & -2 & 0\end{pmatrix}.$$

The eigenvalues of $\boldsymbol{A}_0$ are

$$\begin{aligned} e_1 &= -\gamma\\ e_2 &= -\gamma + \sqrt{\gamma^2 - 4\omega_0^2}\\ e_3 &= -\gamma - \sqrt{\gamma^2 - 4\omega_0^2}. \end{aligned} \tag{26}$$

Equation (15) therefore takes the form of an evolution equation of higher moments,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\langle x^2\rangle\\ \langle xy\rangle\\ \langle y^2\rangle\end{pmatrix} = \boldsymbol{T}\begin{pmatrix}\langle x^2\rangle\\ \langle xy\rangle\\ \langle y^2\rangle\end{pmatrix} \tag{27}$$

where

$$\boldsymbol{T} = \begin{pmatrix} 0 & 2 & 0 \\ \frac{\alpha^2\omega_0^2(\gamma^2c_1-\gamma c_2+2\omega_0^2c_1)}{2\gamma^2}-\omega_0^2 & \frac{\alpha^2(\gamma^3c_1-2\omega_0^2\gamma c_1-2\omega_0^2c_2-\gamma^2c_2)}{2\gamma^2}-\gamma & 1-\frac{\alpha^2(\gamma^2+2\omega_0^2)c_1}{2\gamma^2} \\ \frac{\alpha^2\omega_0^2(\gamma c_2-\gamma^2c_1-\omega_0^2c_1)}{\gamma} & \frac{\alpha^2(\gamma^2c_2+\omega_0^2c_2-\gamma^3c_1+3\omega_0^2\gamma c_1)}{\gamma}-2\omega_0^2 & \frac{\alpha^2(\gamma^2+\omega_0^2)c_1}{\gamma}-2\gamma \end{pmatrix} \tag{28}$$

where $c_1 = \langle\xi^2\rangle$ and $c_2 = \langle\xi\dot{\xi}\rangle$.

What follows as a consequence of equation (27) is the shifting of eigenvalues of the unperturbed oscillator. The eigenvalue which corresponds to e_1 of the unperturbed case now becomes

$$-\gamma - \alpha^2\left\{\frac{\gamma}{2}c_1 + \frac{\omega_0^2}{4\omega_0^2-\gamma^2}c_1 - \frac{1}{2}c_2\right\}$$

to second order in α. Thus the damping of energy of the unperturbed oscillator gets enhanced beyond a critical value depending on the positivity of the term included in the parenthesis of the last expression. In the negative region the term acts as a gain term leading to a suppression of dissipation of energy of the oscillator.

In contrast to this case of adiabatic fluctuations, fast fluctuations make the oscillator unstable energy-wise due to the fluctuations in the force that have twice the frequency of the oscillator. Addition of a damping term, however, may result in stability below a certain critical value and unstability above it. This result has been particularly relevant in establishing the Kubo relation in chaotic dynamics [20].

4.2. *Diffusion in a turbulent fluid*

As a next application we consider a diffusive process in a fluid in motion described by

$$\frac{\partial n}{\partial t} = -\nabla\cdot(n\boldsymbol{v}) + D\nabla^2 n. \tag{29}$$

Here $n(\boldsymbol{r},t)$ is the number of 'probe particles' per unit volume and $\boldsymbol{v}(\boldsymbol{r},t)$ is the velocity of the moving fluid. If turbulence sets in then $\boldsymbol{v}(\boldsymbol{r},t)$ becomes a stochastic function of $\boldsymbol{r}$ and t. The problem has been addressed by numerous workers over several decades. To quote a representative few of them we refer to [14].

For the present problem we consider $\boldsymbol{v}(\boldsymbol{r},t)$ as an adiabatic stochastic process. The problem then is to find an average $n(\boldsymbol{r},t)$ for the given initial condition

$$n(\boldsymbol{r},0) = \delta(\boldsymbol{r}).$$

We consider the turbulence to be weak and slow and without any loss of generality assume

$$\langle\boldsymbol{v}(\boldsymbol{r},t)\rangle = 0.$$

Equation (29) is of the form (3), provided the matrix $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$ correspond to

$$\boldsymbol{A}_0 = D\nabla^2 \qquad \text{and} \qquad \alpha\boldsymbol{A}_1 = \nabla\cdot\boldsymbol{v}.$$

The symbol ∇, as usual, acts on every functions of $\boldsymbol{r}$ that appears to the right of it. Equation (8) then takes the following form:

$$\frac{\partial}{\partial t}\langle n(\boldsymbol{r},t)\rangle = D\nabla^2\langle n(\boldsymbol{r},t)\rangle + \int_0^t \mathrm{d}\tau\,\left\langle\nabla\cdot\boldsymbol{v}(\boldsymbol{r},t)\,\mathrm{e}^{\tau D\nabla^2}\nabla\cdot\boldsymbol{v}(\boldsymbol{r},t-\tau)\,n(\boldsymbol{r},t-\tau)\right\rangle. \tag{30}$$

We take the Fourier transform in space of the last part of equation (30) to obtain

$$\frac{\partial}{\partial t}\langle n(\boldsymbol{k},t)\rangle = -Dk^2\langle n(\boldsymbol{k},t)\rangle - (2\pi)^{-3}\sum_{i,j}\int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle v_i(\boldsymbol{q},t)\, v_j(\boldsymbol{q}',t-\tau)\rangle \times n(\boldsymbol{k},t-\tau)(q_i+q_i'+k_i)\mathrm{e}^{-\tau D(\boldsymbol{k}+\boldsymbol{q}')^2}(q_j'+k_j)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{q}'. \tag{31}$$

Expanding $v_j(\boldsymbol{q}',t-\tau)\, n(\boldsymbol{k},t-\tau)$ as a Taylor series and integrating over τ as before we arrive at

$$\frac{\partial}{\partial t}\langle n(\boldsymbol{k},t)\rangle = -Dk^2\langle n(\boldsymbol{k},t)\rangle - (2\pi)^{-3}\sum_{i,j}\sum_n(-1)^n \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle v_i(\boldsymbol{q},t)\frac{1}{\left[D(\boldsymbol{k}+\boldsymbol{q}')^2\right]^{n+1}}\partial_t^n[v_j(\boldsymbol{q}',t)\, n(\boldsymbol{k},t)]\right\rangle \times(q_i+q_i'+k_i)(q_j'+k_j)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{q}'. \tag{32}$$

Taking into account the following properties of the stochastic function $\boldsymbol{v}(\boldsymbol{q},t)$:

$$\begin{aligned}\langle v_i(\boldsymbol{q},t)\, v_j(\boldsymbol{q}',t)\rangle &= \delta(\boldsymbol{q}+\boldsymbol{q}')(2\pi)^{3/2}C_{ij}(\boldsymbol{q})\\ \langle v_i(\boldsymbol{q},t)\, \dot{v}_j(\boldsymbol{q}',t)\rangle &= \delta(\boldsymbol{q}+\boldsymbol{q}')(2\pi)^{3/2}C_{ij}'(\boldsymbol{q})\end{aligned} \tag{33}$$

and imposing the adiabatic following approximation that $v_j(\boldsymbol{q},t-\tau)\, n(\boldsymbol{k},t-\tau)$ varies much slowly in the time scale of D^{-1} we obtain

$$\frac{\partial}{\partial t}\langle n(\boldsymbol{k},t)\rangle = \left\{-Dk^2 - (2\pi)^{-3/2}\sum_{i,j}\int_{-\infty}^{\infty}C_{ij}(\boldsymbol{q})\, k_i(k_j-q_j)\frac{1}{D(\boldsymbol{k}-\boldsymbol{q})^2}\mathrm{d}\boldsymbol{q} + (2\pi)^{-3/2}\sum_{i,j}\int_{-\infty}^{\infty}C_{ij}'(\boldsymbol{q})\, k_i(k_j-q_j)\frac{1}{D^2(\boldsymbol{k}-\boldsymbol{q})^4}\mathrm{d}\boldsymbol{q} - (2\pi)^{-3/2}\sum_{i,j}\int_{-\infty}^{\infty}C_{ij}(\boldsymbol{q})\, k_i(k_j-q_j)\frac{k^2}{D(\boldsymbol{k}-\boldsymbol{q})^4}\mathrm{d}\boldsymbol{q}\right\}. \tag{34}$$

The effect of adiabatic stochasticity in the motion of the fluid thus essentially is to recover a renormalized diffusion coefficient $D(k)$ of 'test' particles in the following form:

$$D(k) = D + (2\pi)^{-3/2}\sum_{i,j}\int_{-\infty}^{\infty}C_{ij}(\boldsymbol{q})\, k_i(k_j-q_j)\frac{1}{Dk^2(\boldsymbol{k}-\boldsymbol{q})^2}\mathrm{d}\boldsymbol{q} - (2\pi)^{-3/2}\sum_{i,j}\int_{-\infty}^{\infty}C_{ij}'(\boldsymbol{q})\, k_i(k_j-q_j)\frac{1}{D^2k^2(\boldsymbol{k}-\boldsymbol{q})^4}\mathrm{d}\boldsymbol{q} + (2\pi)^{-3/2}\sum_{i,j}\int_{-\infty}^{\infty}C_{ij}(\boldsymbol{q})\, k_i(k_j-q_j)\frac{1}{D(\boldsymbol{k}-\boldsymbol{q})^4}\mathrm{d}\boldsymbol{q}. \tag{35}$$

Hence equation (34) reduces to

$$\frac{\partial}{\partial t}\langle n(\boldsymbol{k},t)\rangle = -D(k)\, k^2\langle n(\boldsymbol{k},t)\rangle \tag{36}$$

a normalized diffusion equation for isotropic turbulence. In the absence of any detailed knowledge about the stochastic properties embedded in $C_{ij}(\boldsymbol{q})$ and $C_{ij}'(\boldsymbol{q})$ it is difficult to proceed further. Nevertheless, in the limit of small $\boldsymbol{q}$ one might expect some interesting behaviour as has been observed in the case of rapid fluctuations.

It is well known that the long wavelength fast fluctuations are insufficiently damped by the viscosity, which appears as a parameter in the correlation function of the incompressible fluids, which ensures the existence of a finite τ_c. This causes long-time tails in the correlation functions. As van Kampen [14] emphasized the stochastic description in terms of an average $\langle n\rangle$ ceases to become meaningful in these cases. Since the present treatment of slow fluctuations is free from explicit correlation functions and we deal only with averages, such pathological problems of long time tails or memory need not trouble us to that extent. This leads us to believe that the average description remains more meaningful in such cases.

5. Conclusion

In this paper we have developed a method for the treatment of weak but adiabatically slow stochastic processes. Based on the 'adiabatic following' approximation we recast a class of linear stochastic differential equations with multiplicative noise into a differential equation for the average solution. This has been carried out on the basis of an expansion in $\alpha|\mu|^{-1}$, where α is the size of the fluctuation and $|\mu|^{-1}$ refers to the time scale of evolution of the unperturbed system. The result differs significantly from the corresponding treatment of weak and rapid fluctuations which relies on the expansion in $\alpha\tau_c$, where τ_c is the autocorrelation time of the fluctuations [14]. It is also necessary to emphasize that in the present work no *a priori* assumption on the nature of noise in $\boldsymbol{A}_1(t)$ (such as $\boldsymbol{A}_1(t)$ is a Gaussian random process, etc, which has received so much attention in the literature) has been made. Although in our applications we have dealt with classical and linear problems, the theory can be extended to quantum mechanical and nonlinear problems as well. We hope to address such issues in future communications.

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